Differentiation of Single-Variable Functions

1. Definition: We say a function $f:(a,b)\to\mathbb{R}$ is differentiable at $x_0\in(a,b)$ if $f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists.

Notice $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=L \iff \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)-Lh}{h}=0$. Rewrite with $x=x_0+h$ yielding $\lim_{x\to x_0}\frac{f(x)-f(x_0)-L(x-x_0)}{x-x_0}=0$. Notice the linear function $f(x_0)+L(x-x_0)$ has a graph of $y=f(x_0)+L(x-x_0)$ which is a linear approximation. We define the error term $F(x)=f(x)-(f(x_0)+L(x-x_0))$. Then $\lim_{x\to x_0}\frac{F(x)}{x-x_0}=0$.

2. Example: $f(x)=1+x+x^2$

2.1. $x_0=0$

Here $L=1$ so our linear approx. is $1+L(x-0)=1+Lx=1+x$, then $F(x)=x^2$ and $\frac{x^2}{x}\to 0$ as $x\to 0$.

2.2. $x_0=1$

Here $L=3$ so our linear approx. is $3+L(x-1)=3+3x-3=3x$, then $F(x)=1-2x+x^2=(x-1)^2$, and $\frac{(x-1)^2}{x-1}\to 0$ as $x\to 1$.

3. Theorem: If f is differentiable at x_0 then f is continuous at x_0.

3.3. Proof:

Notice $f(x)=f(x_0)+L(x-x_0)+F(x)$ and we know $\lim_{x\to x_0}\frac{F(x)}{x-x_0}=0$, then $\lim_{x\to x_0}(f(x)-f(x_0))=\lim_{x\to x_0}(L(x-x_0)+F(x))=\lim_{x\to x_0}F(x)=\lim_{x\to x_0}\left(\frac{F(x)}{x-x_0}(x-x_0)\right)$.

4. Proposition: Assume f, g that are differentiable at x_0, then

4.4. (a) $(\alpha f + g)'(x_0) = \alpha f'(x_0) + g'(x_0)$

4.5. (b) $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$

4.6. (c) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$

5. Theorem: Let $g : (c, d) \to (a, b)$, $x_0 \in (c, d)$, and $f : (a, b) \to \mathbb{R}$. Let $y_0 = g(x_0) \in (a, b)$. Assume g is diff. at x_0 and f is diff. at y_0. Then $f\ \ g$ is diff. at x_0 and $(f\ \ g)'(x_0) = f'(y_0) \cdot g'(x_0)$.

5.7. Proof:

We have $f(y) = f(y_0) + L(y - y_0) + F(y)$, with $\lim_{y \to y_0} \frac{F(y)}{y - y_0} = 0$, and $g(x) = y_0 + (x - x_0) + G(x)$, with $\lim_{x \to x_0} \frac{G(x)}{x - x_0} = 0$, where $y_0 = g(x_0)$. We want to show $\lim_{x \to x_0} \frac{f(g(x)) - f(y_0)}{x - x_0} = L$.

Consider $\frac{f(g(x)) - f(y_0) - L(x - x_0)}{x - x_0}$. Using the expansion of f at y_0, $f(g(x)) - f(y_0) = L(g(x) - y_0) + F(g(x))$, so $f(g(x)) - f(y_0) - L(x - x_0) = L(g(x) - y_0 - (x - x_0)) + F(g(x))$. We expand g, $g(x) - y_0 - (x - x_0) = G(x)$, so the numerator is $F(g(x)) + L\,G(x)$. Dividing by $x - x_0$, $\frac{F(g(x))}{x - x_0} + L\frac{G(x)}{x - x_0}$. The second term goes to 0 since $\lim_{x \to x_0} \frac{G(x)}{x - x_0} = 0$. For the first term, we rewrite $\frac{F(g(x))}{x - x_0} = \frac{F(g(x))}{g(x) - y_0} \cdot \frac{g(x) - y_0}{x - x_0}$. As $x \to x_0$, we have $g(x) \to y_0$, so $\frac{F(g(x))}{g(x) - y_0} \to 0$ and $\frac{g(x) - y_0}{x - x_0} = \ + \frac{G(x)}{x - x_0} \to$.

Thus $\lim_{x \to x_0} \frac{F(g(x))}{x - x_0} = 0$ and so $\lim_{x \to x_0} \frac{f(g(x)) - f(y_0) - L(x - x_0)}{x - x_0} = 0$. This implies $\lim_{x \to x_0} \frac{f(g(x)) - f(y_0)}{x - x_0} = L$, so $f\ \ g$ is differentiable at x_0 and in particular $(f\ \ g)'(x_0) = f'(y_0)g'(x_0)$.